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First-return integrals

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Abstract

Properties of first-return integrals of real functions defined on the unit interval are explored. In particular, first-return integrals are shown to be continuous but not absolutely continuous.
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1. Notation, background, and purpose

We shall use $\mathbb{I}$ to denote the unit interval $[0,1]$ and shall be dealing with real-valued functions defined on $\mathbb{I}$. Underlying all our subsequent definitions is the notion of what we call a *trajectory* on an interval $J \subset \mathbb{I}$. A *trajectory* on J is any sequence $\bar{x} \equiv \{x_n\}$ of distinct points in J, whose range is dense in J. If $J = \mathbb{I}$ we usually refer to a trajectory on $\mathbb{I}$ as simply a trajectory. Any countable dense set $S \subset J$ is called a *support set on J* and, of course, any enumeration of S becomes a trajectory on J. For a given trajectory

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$\bar{x} = \{x_n\}$ and an interval H, we let $r(\bar{x}, H)$ denote the first x_n that belongs to H. When the trajectory $\bar{x}$ is understood, we set $r(H) = r(\bar{x}, H)$. For $x \in \mathbb{I}$ and $\rho > 0$ we let $B_\rho(x) = \{y \in \mathbb{I}\colon |y - x| < \rho\}$ and we use $\lambda(T)$ to denote the Lebesgue measure of a measurable set T. Finally, if f is Lebesgue integrable on a set T, we use both $(L)\int_T f$ and $\int_T f$ to denote the Lebesgue integral of f over T.

Definition 1.1. Let $f : \mathbb{I} \to \mathbb{R}$, let $\bar{t}$ be a trajectory in $\mathbb{I}$, and let H be a subinterval of $\mathbb{I}$. We say that f is *first-return integrable with respect to $\bar{t}$ on H* if there is a finite number A such that the following condition holds: for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition $\mathcal{P} = (x_0 < x_1 < \cdots < x_n)$ of H of mesh $\|\mathcal{P}\| < \delta$ we have

$$\left|\left(\sum_{i=1}^{n} f\big(r\big(\bar{t}, [x_{i-1}, x_i]\big)\big)(x_i - x_{i-1})\right) - A\right| < \varepsilon.$$

In this case we write $(\mathrm{fr}[\bar{t}])\text{-}\int_H f = A$ and call this the first-return integral based on $\bar{t}$ of the function f over H. If the trajectory $\bar{t}$ is understood, we simply denote this integral as $(\mathrm{fr})\text{-}\int_H f$. We shall often specify a partition $\mathcal{P}$ by referring to its partition intervals, instead of the partition points which determine those intervals. Thus, we could rewrite the above displayed inequality as

$$\left|\left(\sum_{J \in \mathcal{P}} f\big(r(\bar{t}, J)\big)|J|\right) - A\right| < \varepsilon,$$

where $|J|$ denotes the length of the partition interval J. We shall also find it convenient at times to let $fr(f, \bar{t}, \mathcal{G}) \equiv \sum_{J \in \mathcal{G}} f(r(\bar{t}, J))|J|$, when $\mathcal{G}$ is any finite collection of non-overlapping intervals. Next, if f is Lebesgue integrable on $\mathbb{I}$, we say that a trajectory $\bar{t}$ *first return yields* (or simply *yields*) *the Lebesgue integral of f on* $\mathbb{I}$ if for every subinterval H of I we have $(\mathrm{fr}[\bar{t}])\text{-}\int_H f = (L)\int_H f$.

It was shown in [1] that if $\bar{t}$ first-return yields the Lebesgue integral of f on $\mathbb{I}$, then for each measurable subset S of $\mathbb{I}$ we have that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\left(\sum_{J \in \mathcal{P}} f\big(r(\bar{t}, J)\big)\lambda(J \cap S)\right) - (L)\int_S f\right| < \varepsilon,$$

whenever $\mathcal{P}$ is a partition of $\mathbb{I}$ with mesh less than δ.

Definition 1.2. Let $x \in I$ and let $\bar{x} = \{x_n\}$ be a fixed trajectory. The *first-return route to x*, is the sequence $\{w_k(\bar{x}, x)\}_{k=1}^{\infty}$ (or more simply $\{w_k(x)\}_{k=1}^{\infty}$ when $\bar{x}$ is understood), defined recursively via

$$w_1(x) = x_0,$$
$$w_{k+1}(x) = \begin{cases} r(\bar{x}, B_{|x - w_k(x)|}(x)), & \text{if } x \neq w_k(x), \\ w_k(x), & \text{if } x = w_k(x). \end{cases}$$

We say that f is *first-return recoverable with respect to* $\bar{x}$ at x, or that $\bar{x}$ *recovers f at x* provided that

$$\lim_{k \to \infty} f\big(w_k(x)\big) = f(x).$$

In [1] close relationships were established between Definitions 1.1 and 1.2. In particular, it was shown there that if a trajectory $\bar{t}$ yields the Lebesgue integral of a function f on $\mathbb{I}$, then $\bar{t}$ recovers f almost everywhere in $\mathbb{I}$. In general the converse is not true for a Lebesgue integrable function; however, it was shown in [1] that a trajectory $\bar{t}$ recovers a *bounded* function f a.e. in $\mathbb{I}$ if and only if it yields the Lebesgue integral of f on $\mathbb{I}$. The purpose of this present work is to explore what can be said in the general (unbounded) case.

2. The continuity of first-return integrals

In this section we shall show that a first-return integral is a continuous function. We begin with a few elementary lemmas.

Lemma 2.1. *Let $f:\mathbb{I}\to\mathbb{R}$, suppose f is first return integrable on I with respect to a trajectory $\bar{t}$ and let H be a subinterval of $\mathbb{I}$. Then f is first return integrable on H with respect to $\bar{t}$.*

Proof. Note that since $\bar{t}$ is a trajectory on $\mathbb{I}$, its restriction to H is a trajectory on H and we could denote this restricted trajectory on H by $\bar{s}$, but since for each interval $J\subset H$ we have $r(\bar{s},J)=r(\bar{t},J)$, we shall simply continue to use $\bar{t}$ instead of $\bar{s}$.

Now, suppose that f is not first-return integrable on H with respect to $\bar{t}$. Then there is an $\varepsilon_0>0$ such that for every $\delta>0$ there are two partitions $\mathcal{Q}(\delta)$ and $\mathcal{R}(\delta)$ of H with $\|\mathcal{Q}(\delta)\|<\delta$ and $\|\mathcal{R}(\delta)\|<\delta)$ such that

$$\left|\sum_{J\in\mathcal{Q}(\delta)} f\bigl(r(\bar{t},J)\bigr)|J|-\sum_{J\in\mathcal{R}(\delta)} f\bigl(r(\bar{t},J)\bigr)|J|\right|>\varepsilon_0.$$

However, since f is first return integrable on $\mathbb{I}$ with respect $\bar{t}$, there is a $\delta_0>0$ such that for any two partitions $\mathcal{Q}$ and $\mathcal{R}$ of $\mathbb{I}$ with $\|\mathcal{Q}\|<\delta_0$ and $\|\mathcal{R}\|<\delta_0$ we have

$$\left|\sum_{J\in\mathcal{Q}} f\bigl(r(\bar{t},J)\bigr)|J|-\sum_{J\in\mathcal{R}} f\bigl(r(\bar{t},J)\bigr)|J|\right|<\varepsilon_0.$$

Now, augment each of $\mathcal{Q}(\delta_0)$ and $\mathcal{R}(\delta_0)$ with the same collection of finitely many points from $I\setminus H$ so that the resulting partitions of $\mathcal{Q}$ and $\mathcal{R}$ of $\mathbb{I}$ have mesh less than δ_0. Then,

$$\begin{aligned}&\left|\sum_{J\in\mathcal{Q}} f\bigl(r(\bar{t},J)\bigr)|J|-\sum_{J\in\mathcal{R}} f\bigl(r(\bar{t},J)\bigr)|J|\right|\\&\quad=\left|\sum_{J\in\mathcal{Q}(\delta_0)} f\bigl(r(\bar{t},J)\bigr)|J|-\sum_{J\in\mathcal{R}(\delta_0)} f\bigl(r(\bar{t},J)\bigr)|J|\right|>\varepsilon_0\end{aligned}$$

and this contradiction completes the proof. □

Thus, the above lemma establishes the existence of first-return integrals over subintervals, and the next lemma illustrates a type of uniformity of approximation of these intervals via first-return sums.

Lemma 2.2. *Suppose that f is first-return integrable with respect to the trajectory $\bar{x}$ on $\mathbb{I}$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each interval $I \subseteq \mathbb{I}$, if $\mathcal{P}$ is a δ-fine partition of I, then*

$$\left| \sum_{J \in \mathcal{P}} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_I f \right| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be such that if $\mathcal{T}$ is a δ-fine partition of $\mathbb{I}$; then

$$\left| \sum_{J \in \mathcal{T}} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_{\mathbb{I}} f \right| < \varepsilon/2.$$

Next, let $I = [a, b] \subseteq \mathbb{I}$ and let $\mathcal{P}$ be a δ-fine partition of I. Let $I^- = [0, a]$, $I^+ = [b, 1]$, and let $\mathcal{P}^-$ and $\mathcal{P}^+$ be partitions of I^- and I^+, respectively, so fine that each has mesh less than δ and

$$\left| \sum_{J \in \mathcal{P}^-} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_{I^-} f \right| < \varepsilon/4 \quad \text{and}$$

$$\left| \sum_{J \in \mathcal{P}^+} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_{I^+} f \right| < \varepsilon/4.$$

Then

$$\begin{aligned} &\left| \sum_{J \in \mathcal{P}} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_I f \right| \\ &\quad = \left| \left(\sum_{J \in \mathcal{P}^- \cup \mathcal{P} \cup \mathcal{P}^+} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_{[0,1]} f \right) - \left(\sum_{J \in \mathcal{P}^-} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_{I^-} f \right) \right. \\ &\qquad \left. - \left(\sum_{J \in \mathcal{P}^+} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_{I^+} f \right) \right| < \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon. \qquad \square \end{aligned}$$

Clearly this lemma may be extended to finite unions of subintervals of $\mathbb{I}$:

Lemma 2.3. *Suppose that f is first-return integrable with respect to the trajectory $\bar{x}$ on $\mathbb{I}$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each union H of finitely many non-overlapping subintervals of I, if $\mathcal{P}$ is a δ-fine partition of H, then*

$$\left| \sum_{J \in \mathcal{P}} f\big(r(J)\big)|J| - (\mathrm{fr})\text{-}\int_H f \right| < \varepsilon.$$

Theorem 2.1. *Suppose that f is first-return integrable with respect to the trajectory $\bar{x}$ on $\mathbb{I}$. Then the function $F(x) = (\mathrm{fr})\text{-}\int_{[0,x]} f$ is continuous on $\mathbb{I}$.*

Proof. Suppose that F is not continuous at some point $p \in \mathbb{I}$. Thus, there is an $\varepsilon > 0$ and a sequence $\{p_n\}$ converging to p such that $|F(p_n) - F(p)| > \varepsilon$. Without loss of generality we may assume that $F(p) = 0$, that $\{p_n\}$ is a strictly decreasing sequence, and that

$$F(p_n) - F(p) = F(p_n) > \varepsilon. \tag{1}$$

Let δ be the positive number corresponding to $\varepsilon/4$ guaranteed by Lemma 2.2. Choose n_1 so that $p_{n_1} - p < \delta$. Letting $r_1 = r([p, p_{n_1}])$ and applying Lemma 2.2 and inequality (1), we have $f(r_1) \cdot (p_{n_1} - p) > 3\varepsilon/4$. Let n_2 be large enough that $p_{n_2} < r_1$ and $f(r_1) \cdot (p_{n_1} - p_{n_2}) > 3\varepsilon/4$. Similarly, by letting $r_2 = r([p, p_{n_2}])$ and applying Lemma 2.2 and inequality (1), we obtain $f(r_2) \cdot (p_{n_2} - p) > 3\varepsilon/4$. Let n_3 be large enough that $p_{n_3} < r_2$ and $f(r_2) \cdot (p_{n_3} - p_{n_2}) > 3\varepsilon/4$. Continuing this process k times, we can obtain a partition $\mathcal{P} = \{p < p_{n_k} < p_{n_{k-1}} < \cdots < p_{n_2} < p_{n_1}\}$ of $[p, p_{n_1}]$ for which

$$\sum_{j \in \mathcal{P}} f\bigl(r(J)\bigr)|J| > k \cdot \varepsilon/4.$$

Since we can do this for all k, this contradicts the fact that f is first-return integrable on $[p, p_{n_1}]$. □

3. A sufficient condition for a first-return integral to be the Lebesgue integral

In the next section we will provide an example of a function f and a trajectory $\bar{t}$ such that, with respect to $\bar{t}$, f is both first-return integrable on $\mathbb{I}$ and a.e. recoverable. Yet the function $F(x) = (\mathrm{fr}[\bar{t}])\text{-}\int_{[0,x]} f$ is not absolutely continuous. The purpose of the present section is to show that if $(\mathrm{fr}[\bar{t}])\text{-}\int_{[0,x]} f$ is absolutely continuous, then f is Lebesgue integrable and $(\mathrm{fr}[\bar{t}])\text{-}\int_I f = (L)\int_I f$.

Lemma 3.1. *Suppose that $f : \mathbb{I} \to \mathbb{R}$ is first return integrable with respect to trajectory $\bar{t}$ and that the function $F(x) = (\mathrm{fr}[\bar{t}])\text{-}\int_{[0,x]} f$ is absolutely continuous. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that if $\mathcal{G} = \{I_1, I_2, \ldots, I_n\}$ is a finite collection of non-overlapping subintervals of $\mathbb{I}$ with $\sum_{i=1}^{n} |I_k| < \delta$, then $|fr(f, \bar{t}, \mathcal{G})| < \varepsilon$.*

Proof. This follows immediately from the definition of absolute continuity and Lemma 2.3. □

Theorem 3.1. *Suppose that $f : \mathbb{I} \to \mathbb{R}$ is first-return integrable and first-return recoverable a.e., both with respect to a trajectory $\bar{t}$, and that the function $F(x) = (\mathrm{fr}[\bar{t}])\text{-}\int_{[0,x]} f$ is absolutely continuous. Then, f is Lebesgue integrable on $\mathbb{I}$ and $(\mathrm{fr})\text{-}\int_{\mathbb{I}} f = (L)\int_{\mathbb{I}} f$.*

Proof. Let $\bar{t}$ be a trajectory with respect to which f is first-return integrable and first-return recoverable a.e. Let $A = (\mathrm{fr})\text{-}\int_{\mathbb{I}} f$. Since f is first return recoverable almost everywhere, Theorem 2.2 in [1] assures that f is measurable. We shall first establish that f is Lebesgue integrable. To achieve a contradiction, assume that f is not Lebesgue integrable. As is standard, we let f^+ and f^- denote the positive and negative parts of f;

i.e. $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\}$. Hence one of f^+ or f^- has Lebesgue integral ∞. Without loss of generality, we assume that the Lebesgue integral of $f^+ = \infty$. Let $\delta > 0$ be such that if $\mathcal{P}$ is a partition of $[0,1]$ with $\|\mathcal{P}\| < \delta$, then $|fr(f, \bar{t}, \mathcal{P}) - A| < 1$. We may also assume that δ is small enough so that it satisfies Lemma 3.1 with $\varepsilon = 1$.

Let $M^+ = f^{-1}([0,\infty))$ and $M_- = f^{-1}((-\infty, 0))$. Using the fact that f is first return recoverable with respect to $\bar{t}$ a.e., we may choose a compact set $N_- \subseteq M_-$ and a positive integer K such that $|M_- \setminus N_-| < \frac{\delta}{2}$ and if $n > K$ and $x \in N_-$ and t_n is in the first return path to x, then $|f(x) - f(t_n)| < 1$. Since f is finite, we may choose a subset of N_- which we also call N_- on which f is bounded below by some constant B and $|M_- \setminus N_-| < \frac{\delta}{2}$ still holds. Since the Lebesgue integral of f^+ is ∞, we may chose a compact set $N^+ \subseteq M^+$ such that f is bounded on N^+ and the Lebesgue integral of f on N^+ is larger than $|B| + |A| + 10$. We may also assume that $|M^+ \setminus N^+| < \frac{\delta}{2}$ and if $n > K$ and $x \in N^+$ and t_n is in the first return path to x, then $|f(x) - f(t_n)| < 1$. Now let $\mathcal{P}$ be a partition generated by an initial finite sequence of $\bar{t}$ so that

- $\|\mathcal{P}\| < \delta$;
- if $I \in \mathcal{P}$, then I intersects at most one of N^+ and N_-;
- if
$$\mathcal{G} = \bigl\{I \in \mathcal{P}\colon\ I \cap \bigl(N^+ \cup N_-\bigr) = \emptyset \text{ or } r(I) \in \{t_1, \ldots, t_K\}\bigr\},$$
then $|\bigcup \mathcal{G}| < \delta$;
- if
$$\mathcal{N}^+ = \bigl\{I \in \mathcal{P}\colon\ I \cap N^+ \neq \emptyset \text{ and } r(I) \notin \{t_1, \ldots, t_K\}\bigr\},$$
then $fr(f, \bar{t}, \mathcal{N}^+) > |B| + |A| + 8$.

Let $\mathcal{N}_- = \{I \in \mathcal{P}\colon\ I \cap N_- \neq \emptyset \text{ and } r(I) \notin \{t_1, \ldots, t_K\}\}$. Then,

$$\begin{aligned} fr(f, \bar{t}, \mathcal{P}) &= fr(f, \bar{t}, \mathcal{G}) + fr\bigl(f, \bar{t}, \mathcal{N}^+\bigr) + fr(f, \bar{t}, \mathcal{N}_-) \\ &> (-1) + \bigl(|B| + |A| + 8\bigr) + (B - 1) \\ &= |A| + 6. \end{aligned}$$

However, this contradicts our choice of δ, completing the proof that f is Lebesgue integrable.

Now, let $A' = (L)\int_{\mathbb{I}} f$, let $0 < \varepsilon < 1$, and let $\delta > 0$ be such that each of the following holds:

- If $\mathcal{P}$ is a partition of $\mathbb{I}$ with $\|\mathcal{P}\| < \delta$, then $|fr(f, \bar{t}, \mathcal{P}) - A| < \frac{\varepsilon}{8}$.
- If $\mathcal{G}$ is a finite collection of non-overlapping subintervals of $\mathbb{I}$ with $|\cup \mathcal{G}| < \delta$, then $|fr(f, \bar{t}, \mathcal{G})| < \frac{\varepsilon}{8}$.
- If $H \subset \mathbb{I}$ with $\lambda(H) < \delta$, then $(L)\int_H |f| < \frac{\varepsilon}{8}$.

Let $M \subseteq \mathbb{I}$ be a compact set such that $\lambda(\mathbb{I} \setminus M) < \frac{\delta}{2}$ and there is a positive integer K such that for each $x \in M$ if $n > K$ and t_n is in the first return path to x, then $|f(x) - f(t_n)| < \frac{\varepsilon}{8}$.

Furthermore, we can assume that f is bounded on M and we let $B > 0$ be a bound on $|f|$ on M. Let $B^* = B + \frac{\varepsilon}{8}$. Next, let $\mathcal{P}$ be a partition of $[0, 1]$ formed by a finite initial sequence of $\bar{t}$ such that

- $\|\mathcal{P}\| < \delta$;
- if $\mathcal{G} = \{I \in \mathcal{P}\colon\ I \cap M \neq \emptyset,\ fr(I) \notin \{t_1, t_2, \dots, t_K\}$ and $\frac{\lambda(I \setminus M)}{|I|} < \frac{\varepsilon}{8 \cdot B^*}\}$, then $\lambda([0, 1] \setminus \bigcup \mathcal{G}) < \frac{\delta}{2}$.

Let $\mathcal{H} = \mathcal{P} \setminus \mathcal{G}$. Now we have that

$$\begin{aligned}
|A - A'| &\leqslant \bigl|A - fr(f, \bar{t}, \mathcal{P})\bigr| + \bigl|fr(f, \bar{t}, \mathcal{P}) - A'\bigr| \\
&\leqslant \frac{\varepsilon}{8} + \bigl|fr(f, \bar{t}, \mathcal{H})\bigr| + \Bigl|fr(f, \bar{t}, \mathcal{G}) - (L) \int\limits_{M \cap \bigcup \mathcal{G}} f\Bigr| + \Bigl|(L) \int\limits_{\mathbb{I} \setminus (M \cap \bigcup \mathcal{G})} f\Bigr| \\
&\leqslant \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \Bigl|\sum_{I \in \mathcal{G}} \Bigl(f\bigl(r(I)\bigr)|I| - (L) \int\limits_{I \cap M} f\Bigr)\Bigr| + \frac{\varepsilon}{8} \\
&= \frac{3\varepsilon}{8} + \Bigl|\sum_{I \in \mathcal{G}} \Bigl(f\bigl(r(I)\bigr)|I| - (L) \int\limits_{I \cap M} f\Bigr)\Bigr| \\
&\leqslant \frac{3\varepsilon}{8} + \Bigl|\sum_{I \in \mathcal{G}} (L) \int\limits_{I \cap M} \bigl(f - f\bigl(r(I)\bigr)\bigr)\Bigr| + \Bigl|\sum_{I \in \mathcal{G}} f\bigl(r(I)\bigr)\lambda(I \setminus M)\Bigr| \\
&\leqslant \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} + \sum_{I \in \mathcal{G}} \bigl|f\bigl(r(I)\bigr)\bigr|\lambda(I \setminus M) \\
&\leqslant \frac{4\varepsilon}{8} + \sum_{I \in \mathcal{G}} B^* \cdot \lambda(I \setminus M) \\
&\leqslant \frac{4\varepsilon}{8} + \sum_{I \in \mathcal{G}} B^* \cdot \frac{\varepsilon}{8 \cdot B^*} \cdot |I| \\
&\leqslant \frac{5\varepsilon}{8}. \qquad \square
\end{aligned}$$

4. An example

Here we shall construct a trajectory $\bar{x} = \{x_n\}$ and a function $f : \mathbb{I} \to \mathbb{R}$ which is 0 for $x \notin \{x_n\}$ such that $\bar{x}$ recovers 0 almost everywhere on $\mathbb{I}$. Moreover, the first-return integral of f with respect to $\bar{x}$ exists but is not 0. This entire section is devoted to this construction. We shall first describe a weighted system of intervals and then use these intervals to define a measure μ on $\mathbb{I}$. The sequence, $\bar{x}$, consists of the centers of these intervals ordered lexicographically, first according to the "stage" of the center and second according to the usual ordering on the real line. The function, f, is defined in such a way that the function value at the center point times the length of the interval is the μ measure of that interval. The argument that $\bar{x}$ recovers the 0 function almost everywhere is probabilistic in nature while the fact that f is first-return integrable with respect to $\bar{x}$ uses the nature of the measure μ. Both of these facts depend on the parameters of the construction.

Let $\{\varepsilon_k\}$ be a monotone decreasing sequence of positive numbers and let n_k be a monotone increasing sequence of natural numbers tending to infinity, respectively. We define a system of intervals inductively where the number of the intervals at stage k will depend on the parameters $n_1, n_2, \dots, n_k$ and the weight we associate with these intervals will depend on $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$. We denote $N_k = 4n_k + 1$.

4.1. Construction of the intervals

First divide $I \equiv [0, 1]$ into N_1 non-overlapping congruent intervals of length $\frac{1}{N_1}$; denote these intervals by $I_1, I_2, \dots, I_{N_1}$. These are the intervals of stage 1.

The intervals of stage 2 are defined as follows. Divide each interval I_j into N_2 non-overlapping congruent intervals of length $\frac{1}{N_1 N_2}$; denote these intervals by $I_{j1}, I_{j2}, \dots, I_{jN_2}$.

Inductively, suppose $I_{\underline{j}}$ has been defined for a finite sequence of indices, $\underline{j} = j_1 j_2 \dots j_k$ and is of length $\frac{1}{N_1 N_2 \dots N_k}$. Divide $I_{\underline{j}}$ into N_{k+1} non-overlapping congruent intervals of length $\frac{1}{N_1 N_2 \dots N_{k+1}}$; denote these intervals by $I_{\underline{j}1}, I_{\underline{j}2}, \dots, I_{\underline{j}N_{k+1}}$. These are the intervals of stage $k+1$. Endpoints (or centers) of intervals of stage $k+1$ which are not endpoints (or centers) of intervals from previous stages will be referred to as endpoints (or centers) of stage $k+1$. We denote the center of $I_{\underline{j}}$ by $c_{\underline{j}}$.

4.2. Construction of the Weights $q_{\underline{j}}$ and the Measure μ

For $\underline{j} = j_1 j_2 \dots j_k$ let

$$r_{\underline{j}} = \begin{cases} 1 & \text{if } j_k \text{ is odd,} \\ 1 - \varepsilon_k & \text{if } j_k \equiv 2 \pmod 4, \\ 1 + \varepsilon_k & \text{if } j_k \equiv 0 \pmod 4. \end{cases}$$

Now define $q(I_{\underline{j}}) = r_{j_1} r_{j_1 j_2} \dots r_{\underline{j}}$ and $\mu(I_{\underline{j}}) = \frac{q(I_{\underline{j}})}{N_1 N_2 \dots N_k}$. This defines a measure, since, for $\underline{j}' = j_1 j_2 \dots j_{k-1}$:

$$\sum_{i=1}^{N_k} \mu(I_{\underline{j}'i}) = \frac{q(I_{\underline{j}'})}{N_1 N_2 \dots N_k} \cdot \sum_{i=1}^{N_k} r_{\underline{j}'i} = \frac{q(I_{\underline{j}'})}{N_1 N_2 \dots N_{k-1}} = \mu(I_{\underline{j}'}).$$

Let $f(c_{\underline{j}}) = q(I_{\underline{j}})$ for the centers of the $I_{\underline{j}}$ and 0 elsewhere. Since all the numbers N_k are of the form $4n_k + 1$, it is easy to see that f is well-defined. Since the first return point of each $I_{\underline{j}}$ is its center, by this choice of f we have $f \circ r(\bar{x}, I_{\underline{j}}) \cdot |I_{\underline{j}}| = \mu(I_{\underline{j}})$.

4.3. Comparing weights

We denote $\delta_k = \frac{1-\varepsilon_k}{1+\varepsilon_k}$. Then $\{\delta_k\}$ is a monotone increasing sequence tending to 1.

It is easy to see that:

(i) For every pair of intervals J_1, J_2 of stage k, if they are subintervals of the same interval J of stage $k-1$, then

$$\frac{q(J_1)}{q(J_2)} \in \left[\delta_k, \frac{1}{\delta_k}\right].$$

(ii) For every $\underline{j} = j_1 j_2 \dots j_k$, if $j_k = 1$ or $j_k = N_k$, then $r_{\underline{j}} = 1$. Hence, if $J_1 \subset J_2$ are two intervals of our construction and they have a common endpoint, then $q(J_1) = q(J_2)$.

(iii) If J_1 and J_2 are two non-overlapping intervals of our construction with a common endpoint and if this endpoint is of stage k, then

$$\frac{q(J_1)}{q(J_2)} \in \left[\delta_k, \frac{1}{\delta_k}\right].$$

(iv) If J_1' and J_2' are two non-overlapping intervals of our construction of stage $k' \geqslant k$ with a common endpoint of stage k, and if J_1 and J_2 are subintervals of stage $k'+1$ of J_1' and J_2', respectively, then

$$\frac{q(J_1)}{q(J_2)} \in \left[\delta_k \delta_{k'+1}, \frac{1}{\delta_k \delta_{k'+1}}\right] \subset \left[\delta_k^2, \frac{1}{\delta_k^2}\right].$$

Now let $J = (a, b)$ be an arbitrary interval (not necessarily of form $I_{\underline{j}}$) with $0 < a < b < 1$. Let k denote the minimal index for which J contains at least one endpoint of an interval of stage k. Let this endpoint be p. Let k_1 (or k_2) denote the minimal index for which (a, p) (or (p, b)) contains at least one endpoint of an interval of stage k_1 (or k_2). Then $k_1, k_2 \geqslant k$. For $\ell \geqslant k_1$ (or $\ell \geqslant k_2$) we denote by $\mathcal{I}_\ell^1$ (or $\mathcal{I}_\ell^2$) the set of all intervals of stage ℓ that intersect (a, p) (or (p, b)). It is easy to see that

(v) It follows from (i) that, if $J_1, J_2 \in \mathcal{I}_{k_i}^i$ for $i = 1$ or $i = 2$, then

$$\frac{q(J_1)}{q(J_2)} \in \left[\delta_{k_i}, \frac{1}{\delta_{k_i}}\right] \subset \left[\delta_k, \frac{1}{\delta_k}\right].$$

(vi) Since $\mathcal{I}_{k_1}^1$ and $\mathcal{I}_{k_2}^2$ contains an interval with endpoint p, it follows from (iii) and (v) that for any $J_1, J_2 \in \mathcal{I}_{k_1}^1 \cup \mathcal{I}_{k_2}^2$:

$$\frac{q(J_1)}{q(J_2)} \in \left[\delta_k^2, \frac{1}{\delta_k^2}\right].$$

(vii) It follows from (vi) and (i) that for any $J_1, J_2 \in \mathcal{I}_{k_1+1}^1 \cup \mathcal{I}_{k_2+1}^2$,

$$\frac{q(J_1)}{q(J_2)} \in \left[\delta_k^4, \frac{1}{\delta_k^4}\right].$$

(viii) The first return point of J is the center of one of the intervals of $\mathcal{I}_{k_1+1}^1 \cup \mathcal{I}_{k_2+1}^2$, hence for any $J_1 \in \mathcal{I}_{k_1+1}^1 \cup \mathcal{I}_{k_2+1}^2$,

$$\frac{f \circ r(\bar{x}, J)}{f \circ r(\bar{x}, J_1)} \in \left[\delta_k^4, \frac{1}{\delta_k^4}\right].$$

Let $\tilde{\mathcal{I}}_\ell^1$ and $\tilde{\mathcal{I}}_\ell^2$ denote the set of intervals of $\mathcal{I}_\ell^1$ and $\mathcal{I}_\ell^2$ inside (a, p) and (p, b), respectively. Since (a, p) contains at least one interval of $\mathcal{I}_{k_1}^1$, we have $|\tilde{\mathcal{I}}_{k_1+1}^1| \geqslant N_{k_1+1} > N_k$ and similarly $|\tilde{\mathcal{I}}_{k_2+1}^2| > N_k$. Clearly

$$\mu\Big(\bigcup \mathcal{I}_{k_1+1}^1 \cup \bigcup \mathcal{I}_{k_2+1}^2\Big) \geqslant \mu(J) \geqslant \mu\Big(\bigcup \tilde{\mathcal{I}}_{k_1+1}^1 \cup \bigcup \tilde{\mathcal{I}}_{k_2+1}^2\Big).$$

From this and (viii) we get

Lemma 4.1. *If for an interval $J = (a, b)$, k is the minimal index for which J contains at least one endpoint of an interval of stage k, then*

$$\frac{f \circ r(\bar{x}, J) \cdot |J|}{\mu(J)} \in \left[\delta_k^4 \cdot \frac{N_k}{N_k + 1}, \frac{1}{\delta_k^4 \cdot \frac{N_k}{N_k+1}}\right].$$

4.4. The first return integral is μ

Fix an interval $I_0 \subset \mathbb{I}$; we show that the first return integral of f relative to $\bar{x}$ exists and

$$\text{(fr)-}\int_{I_0} f = \mu(I_0).$$

Suppose $\varepsilon > 0$ and $k \in \mathcal{N}$ are given and let $\mathcal{P}$ be any partition, sufficiently fine so that ε exceeds the μ measure of the union of the intervals covering the endpoints of the kth stage. Define $\mathcal{P}_1 = \{I \in \mathcal{P}\colon\ I$ contains an endpoint of the kth stage$\}$ and $\mathcal{P}_2 = \mathcal{P} \backslash \mathcal{P}_1$. Then it follows from Lemma 4.1 that

$$\sum_{\mathcal{P}_1} f\big(r(\bar{x}, J)\big)|J| \in \left[0, \varepsilon \cdot \frac{1}{\delta_1^4} \cdot \frac{N_1 + 1}{N_1}\right].$$

For intervals $J \in \mathcal{P}_2$ we have

$$\frac{f(r(\bar{x}, J))|J|}{\mu(J)} \in \left[\delta_k^4 \cdot \frac{N_k}{N_k + 1}, \frac{1}{\delta_k^4} \cdot \frac{N_k + 1}{N_k}\right].$$

Summing over $\mathcal{P}$, we obtain

$$\sum_{J \in \mathcal{P}} f\big(r(\bar{x}, J)\big)|J| \in \left[\delta_k^4 \cdot \frac{N_k}{N_k + 1} \cdot \mu(I_0), \frac{1}{\delta_k^4} \cdot \frac{N_k + 1}{N_k}\mu(I_0) + \varepsilon \cdot \frac{1}{\delta_1^4} \cdot \frac{N_1 + 1}{N_1}\right].$$

As this tends to $\mu(I_0)$ as $k \to \infty$ and $\varepsilon \to 0$, it follows that the first return integral of f with respect to $\bar{x}$ exists and is μ.

In the following subsections we show that there is a choice of the parameters n_k and ε_k such that $\bar{x}$ recovers 0 almost everywhere.

4.5. Remarks on the sequence $w_i(y)$

If $y \in [0, 1]$ is not an endpoint of any interval $I_{\underline{j}}$, then there is a unique sequence $j_1 = j_1(y), j_2 = j_2(y), \ldots$ such that $y \in I_{j_1 j_2 \ldots j_k}$ for every k. We will use the notation $\underline{j} = \underline{j}(y)$

and $I_k(y) = I_{\underline{j}(y)}$, $c_k(y) = c_{\underline{j}}(y)$, $r_k(y) = r_{\underline{j}}(y)$. We denote the minimum of the stages of the two endpoints of $I_k(y)$ by $s_k(y)$.

It is immediate to see that for any $y \in [0,1]$ which is not an endpoint of our construction, $|y - c_k(y)|$ is a decreasing sequence. Since $c_k(y)$ is closer to y than the center of any other interval of stage k, this implies that the sequence $\{w_i(y)\}$ contains all the points $c_k(y)$. Moreover, from the construction of $\{w_i(y)\}$ it follows that all the points $w_i(y)$ of this sequence of stage $k+1$ are either in $I_k(y)$ or in one of its neighbors. Hence for these i's, from (iv) we get

$$\frac{f(w_i(y))}{f(c_{k+1}(y))} \in \left[\delta^2_{s_k(y)}, \frac{1}{\delta^2_{s_k(y)}}\right].$$

Since $s_k(y)$ tends to ∞ as $k \to \infty$, we can see that:

Lemma 4.2. *If y is not the endpoint of any interval of our construction and if* $\lim f(c_k(y))$ *exists, then* $\lim f(w_i(y))$ *exists and the two limits are equal.*

4.6. Recovering zero almost everywhere

We use a probabilistic argument to show that there is a choice of the parameters $\{\varepsilon_k\}$, $\{n_k\}$ such that $\bar{x}$ recovers 0 almost everywhere. By Lemma 4.2 it is enough to show that $\lim f(c_k(y)) = 0$ almost everywhere. Since $f(c_k(y)) = r_1(y) \cdot r_2(y) \cdot \cdots \cdot r_k(y)$, we have to show that for some monotone decreasing sequence $\varepsilon_1, \varepsilon_2, \ldots$ tending to infinity, we have

$$\sum_{k=1}^{\infty} \log r_k(y) = -\infty.$$

For any $0 < \eta < 1$ and $m \in \mathcal{N}$ define the random variable

$$X_{\eta,m} = \begin{cases} 0 & \text{with probability } \frac{2m+1}{4m+1}, \\ \log(1-\eta) & \text{with probability } \frac{m}{4m+1}, \\ \log(1+\eta) & \text{with probability } \frac{m}{4m+1}. \end{cases}$$

Then $\log r_k(y)$ has the same distribution as X_{ε_k,n_k} and from the homogeneity of the definition of $r_k(y) = r_{\underline{j}(y)}$ we see that it is enough to prove the following lemma.

Lemma 4.3. *There are independent random variables X_{ε_k,n_k} for some monotone decreasing sequence $\varepsilon_1, \varepsilon_2, \ldots$ tending to 0 and a monotone increasing sequence of natural numbers $n_1, n_2, \ldots$ tending to infinity such that*

$$\sum_{k=1}^{\infty} X_{\varepsilon_k,n_k} = -\infty$$

with probability 1.

Proof. First we choose an arbitrary monotone decreasing sequence $\eta_1, \eta_2, \ldots$ tending to 0 and a monotone increasing sequence of natural numbers $m_1, m_2, \ldots$ tending to infinity. Let $X_k^1, X_k^2, \ldots$ be independent copies of X_{η_k, m_k}. Since their expected value is negative,

$$\sum_{i=1}^{\infty} X_k^i = -\infty$$

with probability 1. In particular, there is an $a_k \in \mathbb{R}$ such that

$$\operatorname{prob}\bigl(X_k^1 + X_k^2 + \cdots + X_k^n < a_k \text{ for every } n\bigr) > 1 - \frac{1}{2^k}$$

and there is a $b_k \in \mathcal{N}$ such that

$$\operatorname{prob}\bigl(X_k^1 + X_k^2 + \cdots + X_k^{b_k} < -a_{k+1} - k\bigr) > 1 - \frac{1}{2^k}.$$

From the Borel–Cantelli lemma it follows that, with probability 1, if k is large enough then $X_k^1 + X_k^2 + \cdots + X_k^{b_k} < -a_{k+1} - k$ and for every $1 \leqslant n \leqslant b_{k+1}$, $X_{k+1}^1 + X_{k+1}^2 + \cdots + X_{k+1}^n < a_{k+1}$.

Therefore by choosing $\varepsilon_i = \eta_k$ and $n_i = m_k$ for $\sum_{j=1}^{k-1} b_j < i < \sum_{j=1}^{k} b_j$, Lemma 4.3 is verified. $\square$

This then finishes the proof that $\bar{x}$ recovers 0 almost everywhere and therefore establishes the following:

Theorem 4.1. *There exist a function $f : \mathbb{I} \to \mathbb{R}$, a measure μ, and a trajectory $\bar{x} = \{x_n\}$ such that*

(1) $\bar{x}$ *recovers* 0 *almost everywhere and* $f(x) = 0$ *for each* $x \notin \{x_n\}$.
(2) *For each interval* $I \subseteq \mathbb{I}$, $(\mathrm{fr}[\bar{x}])\text{-}\int_I f = \mu(I) \neq 0$.

Note that if f is the function of Theorem 4.1, then Theorem 3.1 assures that the function $F(x) \equiv (\mathrm{fr})\text{-}\int_{[0,x]} f$ is not absolutely continuous.

5. Not every measure can be obtained as a first-return integral

In the previous section we saw that a certain singular measure could be obtained as a first-return integral; we next wish to observe that not every measure can be so obtained. We shall observe a necessary condition for a measure to be obtainable as a first return integral. We begin with a definition.

Definition 5.1. For any closed interval $I = [a, b]$, we let $I^l = [a, (a+b)/2]$ and $I^r = [(a+b)/2, b]$. We say that a measure μ on $\mathbb{I}$ is *balanced* if

$$\lim_{I \to x} \frac{\mu(I^l)}{\mu(I^r)} = 1$$

for μ-a.e. $x \in \mathbb{I}$. As usual, the symbol $I \to x$ indicates the limit taken over closed intervals containing x with lengths tending to 0.

Proposition 5.1. *Let μ be a measure on $\mathbb{I}$ such that there is a function $f:[0,1] \mapsto \mathbb{R}^+$ and a trajectory for which* (fr)-$\int_I f = \mu(I)$ *for every subinterval $I \subset \mathbb{I}$. Then μ is a balanced measure.*

Proof. Suppose μ is a measure on $\mathbb{I}$ for which there exists a function $f:[0,1] \mapsto \mathbb{R}^+$ and a trajectory for which (fr)-$\int_{\mathbb{I}} f = \mu(I)$ for every subinterval $I \subseteq \mathbb{I}$. We wish to show that the set of all x for which $\lim_{I\to x} \frac{\mu(I^l)}{\mu(I^r)} \neq 1$ is of μ measure zero. Suppose there exists a $0 < \alpha < 1$ and a compact set E of positive μ-measure such that for all $x \in E$ there is an arbitrarily short interval $I \ni x$ such that

$$\frac{\mu(I^r)}{\mu(I^l)} \notin [\alpha, 1/\alpha]. \tag{2}$$

If we can show that this situation leads to a contradiction, then we have a complete proof.

From Lemma 2.3 we know that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for each union H of finitely many non-overlapping subintervals, we have that if $\mathcal{P}$ is a δ-fine partition of H, then

$$\left| \sum_{J\in\mathcal{P}} f\bigl(r(J)\bigr)|J| - (\mathrm{fr})\text{-}\int_H f \right| < \varepsilon.$$

From this it follows immediately that for every $0 < \beta < 1$ and $c > 0$ there exists a δ such that for each union H of finitely many non-overlapping subintervals of length at most δ, if $\mu(H) > c$ then for any two subpartitions of H:

$$\frac{\sum_{J\in\mathcal{P}_1} f(r(\bar{t}, J))\lambda(J)}{\sum_{J\in\mathcal{P}_2} f(r(\bar{t}, J))\lambda(J)} \in [\beta, 1/\beta]. \tag{3}$$

Since from any collection of intervals that cover E one can choose a subcollection of disjoint intervals that covers at least half of E, it is enough to show that for every α there exists a β such that if an interval I satisfies (2) then it has subpartitions $\mathcal{P}_1, \mathcal{P}_2$ for which (3) fails.

Let I be an arbitrary interval. Without loss of generality we can assume that $r(\bar{t}, I) \in I^l$. Assume that (3) holds for any two subpartition of I. We denote $f(r(\bar{t}, I^l))|I^l| = f(r(\bar{t}, I))|I|/2 = m_0$, $\mu(I^l) = m_1$, and $\mu(I^r) = m_2$. Then the ratio between any two of the numbers $2m_0$, $m_0 + m_2$, $m_1 + m_2$ is in the interval $[\beta, 1/\beta]$. If β is close enough to 1 then this implies that the ratio between any two of m_0, m_1, m_2 is close to 1; that is, (2) fails. $\square$

As a specific example of a measure on $\mathbb{I}$ which is not balanced, consider a measure μ which is supported on the standard Cantor middle thirds set C. Note that for each $x \in C$ there is a sequence $\{I_{x,n}\}$ of intervals converging to x such that for each n one of the measures $\mu(I_{x,n}^l)$ and $\mu(I_{x,n}^r)$ will be zero and the other non-zero.

6. Open questions

We conclude this paper with some open questions: First, which measures can be obtained from first-return integrals of a non-negative function on $\mathbb{I}$? That is, can one classify the measures μ for which there is a function $f:[0,1] \mapsto \mathbb{R}^+$ and a trajectory such that (fr)-$\int_I f = \mu(I)$ for every subinterval $I \subseteq \mathbb{I}$. For the remaining questions in this paragraph assume that the function f is first-return integrable on $\mathbb{I}$ with respect to the trajectory $\bar{x}$. In Theorem 2.1 we showed that the function $F(x) \equiv$ (fr)-$\int_{[0,x]} f$ is continuous, whereas Theorems 3.1 and 4.1 show that F can fail to be absolutely continuous. Must F be of bounded variation? Does the answer change if we further assume that $\bar{x}$ recovers f a.e.? Must the first-return integrable function f be Lebesgue integrable? We do not know answers to these questions, but as partial insight into the last one, we provide the following:

Proposition 6.1. *If $f:\mathbb{I} \to [0,\infty)$ is first-return recoverable a.e. and first-return integrable, both with respect to the trajectory $\bar{x}$, then f is Lebesgue integrable and $(L)\int_{\mathbb{I}} f \leqslant$ $(\mathrm{fr}[\bar{x}])$-$\int_{\mathbb{I}} f$.*

Proof. For each natural number n, let f_n denoted the truncated function given by $f_n(x) = \min\{f(x), n\}$. Since $\bar{x}$ recovers f a.e., it readily follows that for each n, $\bar{x}$ also recovers f_n a.e. Then, by Theorem 2.2 in [1] it follows that for each n, $\bar{x}$ yields the Lebesgue integral of f_n and thus,

$$(L)\int\limits_{\mathbb{I}} f_n = \big(\mathrm{fr}[\bar{x}]\big)\text{-}\int\limits_{\mathbb{I}} f_n \leqslant \big(\mathrm{fr}[\bar{x}]\big)\text{-}\int\limits_{\mathbb{I}} f,$$

and from this and the Lebesgue Monotone Convergence Theorem, the result follows. □

A final and more open-ended problem is that of defining what might be thought of a THE first-return integral of a function and investigating its properties. For example, if one said that THE first-return integral of f over $\mathbb{I}$ is A if and only if $(\mathrm{fr}[\bar{x}])$-$\int_{\mathbb{I}} = A$ for almost every $\bar{x}$ in the space of sequences in $\mathbb{I}$, what would be the properties of this “integral”?

Reference

[1] M.J. Evans, P.D. Humke, Almost everywhere first-return recovery, Bull. Polish Acad. Sci. Math. 52 (2004) 185–195.